Killara
HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2019 MATHEMATICS EXTENSION 2

Task 4

Date: 25 July 2019

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100 marks

Multiple Choice - Q1 to 10	/10
Question 11	/15
Question 12	/15
Question 13	/15
Question 14	/15
Question 15	/15
Question 16	/15
Total	**/100**

This question paper must not be removed from the examination room.

This assessment task constitutes 30% of the course.

Section I

10 marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section

Answer using multiple-choice answer sheet for questions 1 to 10 (Detach from paper)

1. The reciprocal of $4 + 3i$ is

(A) $\frac{4}{25} + \frac{3}{25}i$

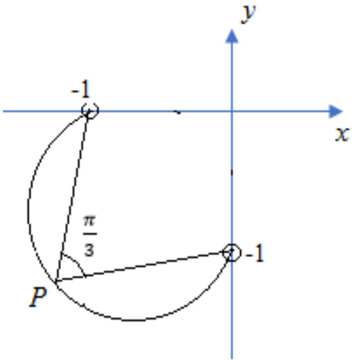
(B) $\frac{4}{5} + \frac{3}{5}i$

(C) $\frac{4}{25} - \frac{3}{25}i$

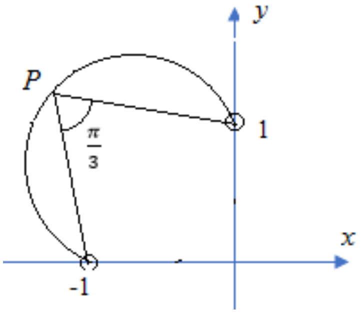
(D) $-\frac{4}{25} + \frac{3}{25}i$

2. Z satisfies $\arg\left(\frac{z+1}{z+i}\right) = -\frac{\pi}{3}$, The locus of P representing Z in the Argand diagram is

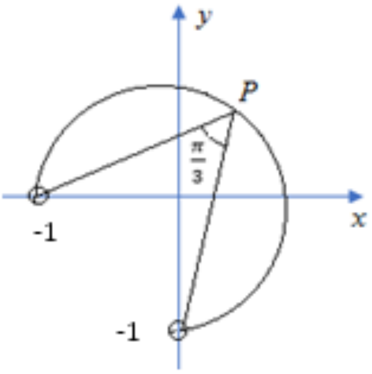
(A)



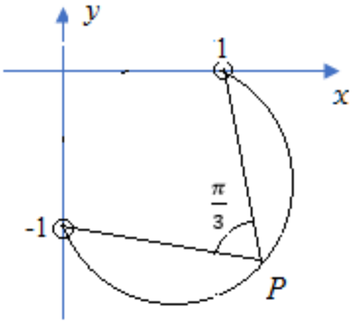
(B)



(C)



(D)



3. Given that $P(x) = x^4 - 5x^2 + 12x + 28$ has an integer that is a double root. $P(x)$ is expressed in terms of real factors as:

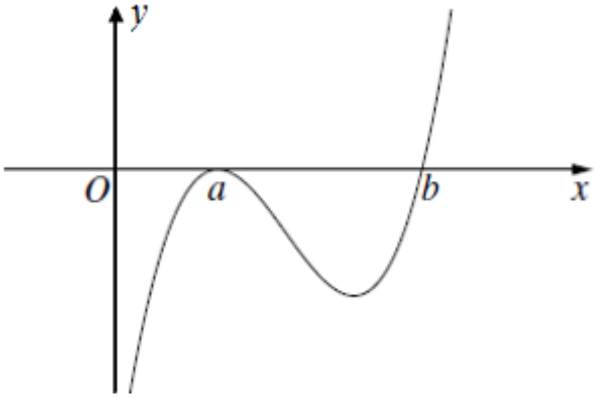
(A) $(x+2)^2(x^2-4x+7)$

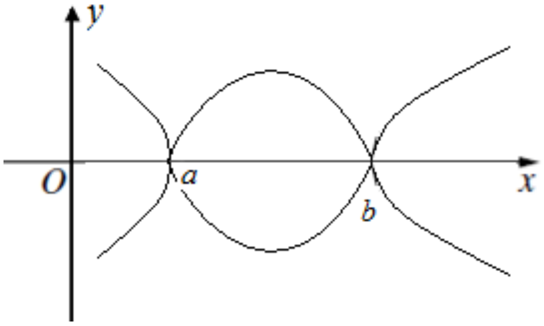
(B) $(x+2)^2(x^2+4x+7)$

(C) $(x-2)^2(x^2+4x+7)$

(D) $(x-1)^2(x^2+4x+28)$

4. The graph of the function $y = f(x)$ is shown.





Which equation best represents the second graph?

(A) $y^2 = |f(x)|$

(B) $y^2 = f(x)$

(C) $y = \sqrt{f(x)}$

(D) $y^2 = f|x|$

5. What is the eccentricity of the ellipse $16x^2 + 25y^2 = 400$

(A) 0.25

(B) 0.36

(C) 0.6

(D) 0.75

6. The equation of a conic with eccentricity $\sqrt{2}$ and asymptotes $y = \pm x$ is:

(A) $xy = 2$

(B) $x^2 - y^2 = 4$

(C) $xy = 1$

(D) $\frac{x^2}{4} - \frac{y^2}{1} = 1$

7. A 200 g mass is swung in a horizontal circle. It completes 5 revolutions in 3 seconds. The circle has a 2 metre diameter.
Which of the following forces is closest to that required to keep the 200g mass moving in this circle?

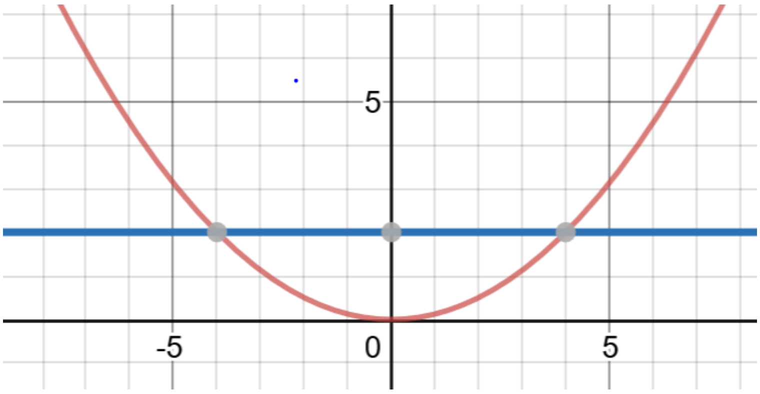
(A) O.5 N

(B) 2.5 N

(C) 10 N

(D) 20 N

8. The volume of the solid generated when the area bounded by $y = 2\ and$ the curve $x^2 = 8y$ is rotated about the line $y = 2$ using the method of slicing (and taking slices perpendicular to the x-axis) is given by:



A) $V = \pi \int_{-4}^{4}(4 - \frac{x^4}{64})dx$

B) $V = \pi \int_{-2}^{2}(4 - \frac{x^4}{64})dx$

C) $V = \pi \int_{-4}^{4}(2 - \frac{x^2}{8})^2 dx$

D) $V = \pi \int_{-2}^{2}(2 - \frac{x^2}{8})^2 dx$

9. Using the recurrence relation $I_n = \int tan^n\, x\ dx = \frac{tan^{n-1}x}{n-1} - I_{n-2}$,

$\int tan^6\, x\ dx$ is equivalent to:

A) $\frac{tan^4 x}{4} - \frac{tan^2 x}{2} + x + c$

B) $\frac{tan^4 x}{4} - \frac{tan^2 x}{2} + x + c$

C) $\frac{tan^6 x}{6} - \frac{tan^4 x}{4} + \frac{tan^2 x}{2} + c$

D) $\frac{tan^5 x}{5} - \frac{tan^3 x}{3} + tan\ x\ - x + c$

10. $\frac{cos\ 4\theta + isin4\theta}{cos\ 2\theta\ -\ isin2\theta}$ simplifies to:

A) $cos\ 2\theta + isin\ 2\theta$

B) $cos\ 6\theta + isin6\theta$

C) $cos\ 2\theta\ -\ isin2\theta$

D) $cos\ 6\theta\ -\ isin6\theta$

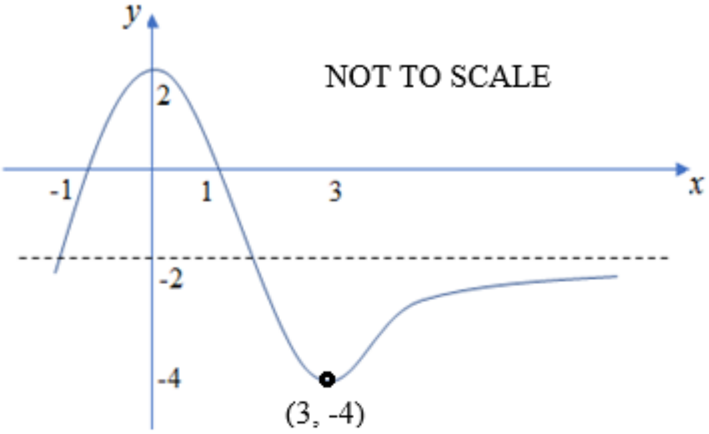
Question 11 (15 marks) Use a NEW Writing Booklet.

(a) (i) Write $2 + 2i$ in the form $r(cos\theta + isin\theta)$. 2

(ii) Hence, or otherwise, find $(2 + 2i)^5$ in the form $a + ib$, where a and b are integers. 2

(b) The diagram below shows the graph of $y = f(x)$.

The line $y = -2$ is an asymptote.



Draw separate **one-third page sketches** of the following.

(i) $y = \frac{1}{f(x)}$ 2

(ii) $|y| = f(|x|)$ 2

(iii) $y = \ln(f(x))$ 2

(c) (i) Express $\frac{3}{(x+1)(x^2+2)}$ in the form $\frac{a}{x+1} + \frac{bx+c}{x^2+2}$, where a, b and c are constants. 2

(ii) Hence find $\int \frac{3}{(x+1)(x^2+2)}\,dx$ 3

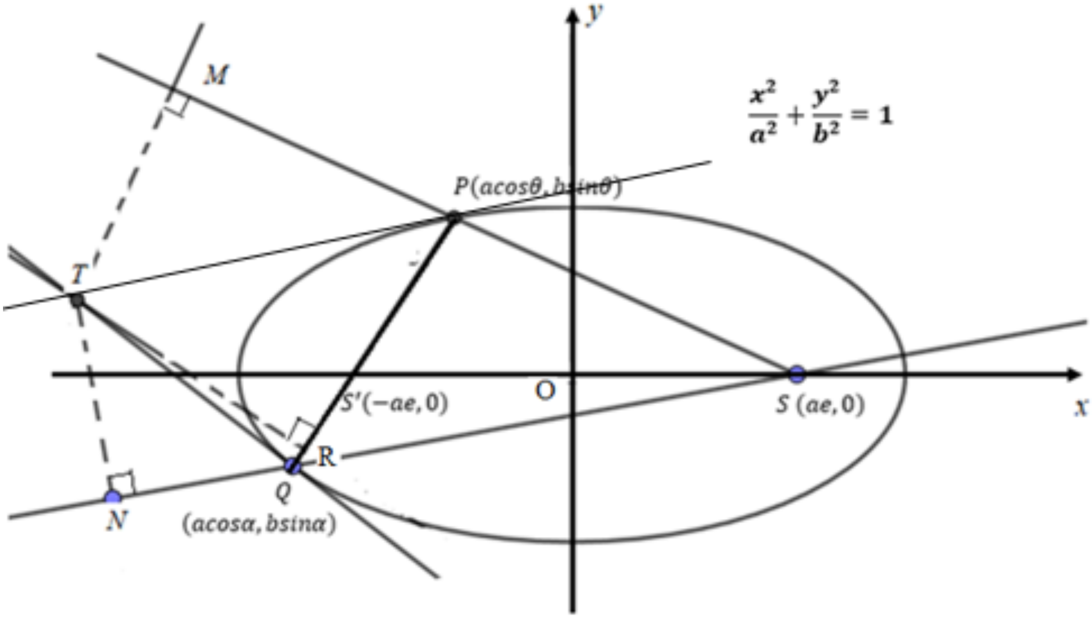
END OF Q11

Question 12 (15 marks) Use a NEW Writing Booklet.

(a) $P(a\,cos\theta, bsin\,\theta)$ and $Q(a\,cos\alpha, bsin\alpha)$ are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. PQ is a focal chord passing through the focus $S'(-ae, 0)$. The tangents to the ellipse at P and Q meet at T.

M, N and R are the feet of the perpendiculars from T to SP, SQ and PQ respectively.

It is given that $TR = \frac{b(1+ecos\theta)}{e\,sin\theta}$, where e is the eccentricity of the ellipse. (You do **NOT** need to prove this)



(i) Write down the x-coordinate of T **1**

(ii) Show that the y-coordinate of T is $\frac{b(e+cos\theta)}{esin\theta}$ **1**

(iii) Show that the equation of SP is **2**

$$(bsin\theta)x + a(e - cos\theta)y \;-\; abesin\theta = 0$$

(iv) Find the length of TM **3**

(v) Hence deduce that SM, PQ and SN are tangents to a circle with its centre at T. **2**

(b) If $z = cos\theta + isin\theta$ then using De Moivre’s theorem it can be shown that: $z^n + \frac{1}{z^n} = 2cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2isin(n\theta)$. **3**

Prove that $cos^6\theta - sin^6\theta = \frac{1}{16}(cos\ 6\theta + 15cos2\theta)$

(c) (i) $Find\ \frac{d}{dx}\left(\frac{ln\ x}{x}\right)$ **1**

(ii) Hence evaluate $\int \frac{1-ln\ x}{xlnx}\ dx$. **2**

END OF Q12

Question 13 (15 marks) Use A NEW Writing Booklet.

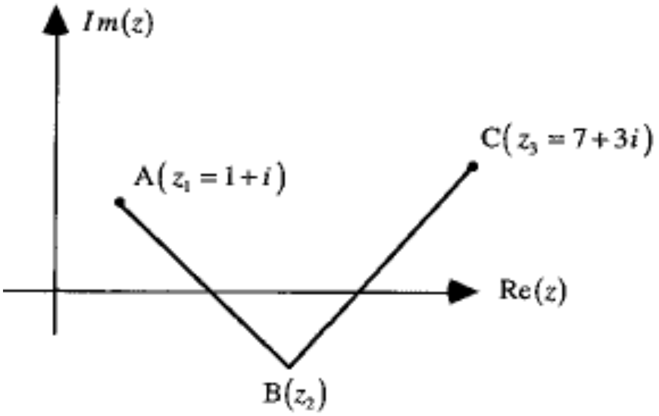
(a) (i) Show that $\cos 2\theta = \frac{1-t^2}{1+t^2}$ where $t = \tan\theta$ **2**

(iii) Hence or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{4}} \frac{4}{5 - 3\cos 2\theta}\, d\theta$$

2

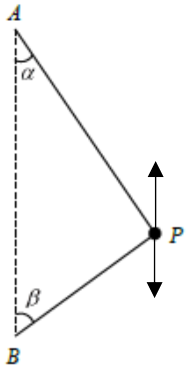
(b)



The points A and C represent the complex numbers $z_1 = 1 + i$ and $z_3 = 7 + 3i$.

Find the complex number z_2 represented by B such that ΔABC is isosceles and right angled at B.

(c) A particle P of mass m spins with angular velocity ω in a circle of radius r, and is suspended by two light inextensible strings making angles from the vertical of α and β, where $0 < \alpha < \beta < \frac{\pi}{2}$. Let A be the point from which the top string is suspended from, and let B (directly below A) be the point where the bottom string is attached.

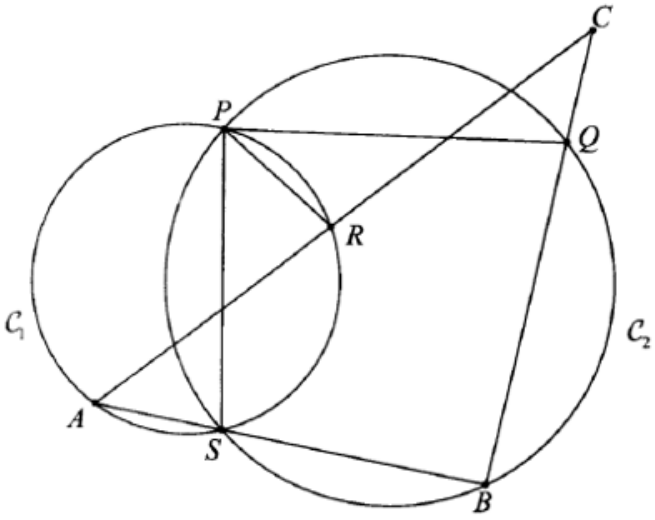


The string AP and BP experiences tensions of T_1 *and* T_2 respectively.

a) Draw all the forces acting on P **1**

b) Prove that if $T_1 > T_2$, then $\omega^2 < \frac{g}{r}\left(\frac{sin\alpha + sin\beta}{cos\alpha - cos\beta}\right)$ **3**

(d) Two circles C_1 and C_2 meet at P and S. Point A and R lie on C_1 and point B and Q lie on C_2. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



(i) Prove that $\angle PRA = \angle PQB$. **2**

(ii) Prove that the points P, R, Q and C are concyclic. **2**

END OF Q13

Question 14 (15 marks) Use a NEW Writing Booklet.

(a) Sketch the curve $y = (x - 2)(6 - x)$ for $x \geq 0$ and find its turning point. 2

(b) The region bounded by the curve y = (x − 2)(6 − x) in the first quadrant and the x-axis is rotated about the y-axis to form a solid. When the region is rotated, the horizontal line segment at height **y sweeps** out an annulus.

(i) Show that the area of the annulus at height y is given by $16\pi\sqrt{4 - y}$. 3

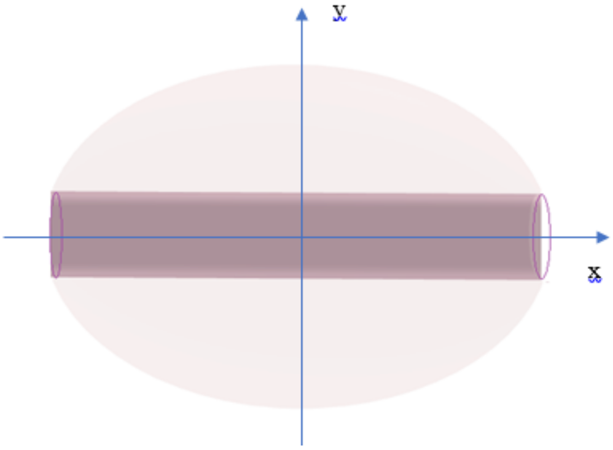
(ii) Find the volume of the solid. 2

(c) (i) Given $I_n = \int_0^3 x^n \sqrt{9 - x^2} dx$, where $n > 1$, prove that 3

$$I_n = \frac{9(n-1)}{n+2} I_{n-2}$$

(ii) Hence evaluate $\int_0^3 x^5 \sqrt{9 - x^2} dx$ 2

(d) A hole of radius 1 unit is bored through the centre, parallel to the major axis of the ellipsoid (football shaped) whose cross section is $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the volume of the remaining solid. 3



END OF Q14

Question 15 (15 marks) Use a NEW Writing Booklet.

(a) Consider the function $f(x) = e^x - e^{-x}$.

(i) Show that $f(x)$ is increasing for all values of x. **1**

(ii) Show that the inverse function is given by **2**

$$f^{-1}(x) = \log_e\left(\frac{x+\sqrt{x^2+4}}{2}\right)$$

(iii) Hence, or otherwise, solve $e^x - e^{-x} = 5$. Give your answer correct to 2 decimal places. **1**

(b) $P(x) = 2x^3 - Ax - 2 = 0$ has roots $\alpha, \beta,$ and γ.

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 = A$ **1**

(ii) Show that $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = A\alpha - \alpha^3$ **2**

(iii) Find the polynomial with the three roots **2**

$$\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \quad \frac{\beta}{\alpha} + \frac{\alpha}{\beta} \;\; and \;\; \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$$

(YOU MAY LEAVE YOUR ANSWER IN UNEXPANDED FORM)

(c) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity $v\ ms^{-1}$.

(i) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where $V\ ms^{-1}$ is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds. **1**

(ii) Find the time T seconds taken for the particle to attain 50% of its terminal velocity, and **3**

(iii) Find the distance fallen in terms of t, v and k. **2**

END OF Q15

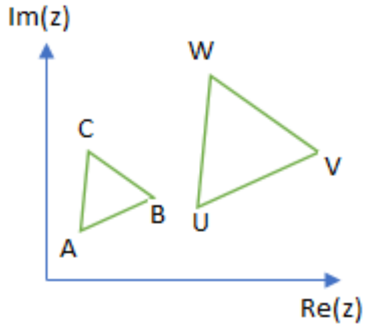
Question 16 (15 marks) Use a NEW Writing Booklet.

(a) (i) In an Argand diagram points A, B, C, U, V and W represent complex numbers a, b, c, u, v and w respectively. 3

Prove that, if the triangles ABC and UVW are directly similar, then

$$aw + bu + cv = av + bw + cu.$$

(Directly similar means that if you go around the triangle in order A, B C and U, V, W then you go around both triangles in the same sense).



(b)

(ii) Show that the triangle ABC is equilateral if and only if 3

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

(i) Sketch a third of a page sized graph of $y = \sqrt{x}$ and indicate on your graph, the region represented by the series

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + ----------- + \sqrt{n}.$$ 1

(c)

(ii) Hence show that $\sqrt{1} + \sqrt{2} + \sqrt{3} + ----- + \sqrt{n} > \frac{2n\sqrt{n}}{3}$ 2

(iii) Hence show that $(4n + 3)\sqrt{n} < (4n + 1)\sqrt{n + 1}$. 2

Find the number of different arrangements of the letters In the word 'PERSEVERE' if:

(i) No 2 'E" are together 2

(ii) Each arrangement must start and end with either 'S' or 'P' with none of the E's together. 2

End of Examination

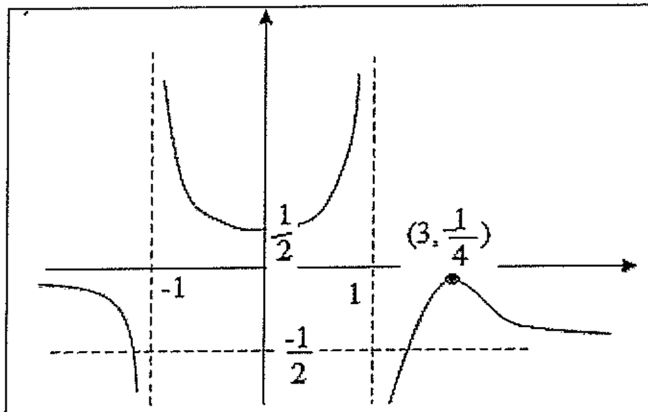
2019-Mathematics Extension 2 Trial HSC

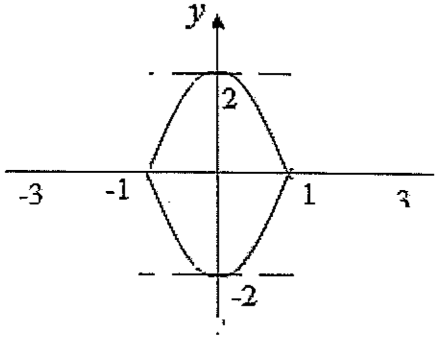
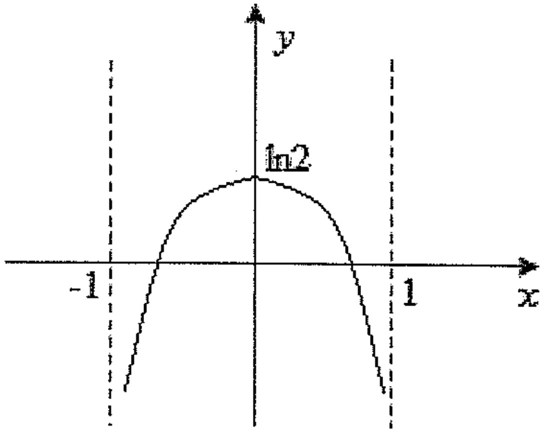
Q1 C	**Q2 C**	**Q3 A**	**Q4 A**	**Q5 C**	**Q6 A**	**Q7 D**	**Q8 C**	**Q9 D**	**Q10 B**

Q11 (a)

(i) $2+2i = 2\sqrt{2}\,(\cos\frac{\pi}{4} + isin\pi/4)$ (ii) $(2+2i)^5 = 32 \times 4\sqrt{2}\,\left(\cos\frac{5\pi}{4} + isin\ \frac{5\pi}{4}\right)$ $= 128\sqrt{2}\,\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$ $= -128 - 128i$	2- correct modulus & argument 2- correct modulus & argument (with working)	Well done

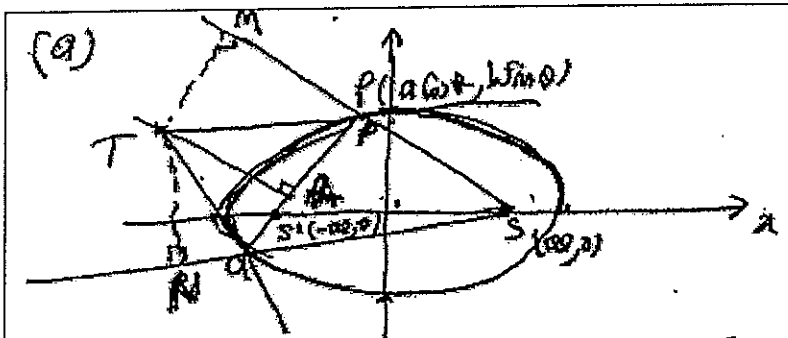
Q11(b)

	2 – correct graph with labelling	Well done

	2 – correct graph with labelling	Only a few got correct ans. Students need to recognise the difference between $\|y\| = f(\|x\|)$ and $\|y\| = \|f(\|x\|)\|$
	2 – correct graph with labelling	Generally well done
(c) (i) $\frac{a}{x+1}+\frac{bx+c}{x^2+2}=\frac{a(x^2+2)+(bx+c)(x+1)}{(x+1)(x^2+2)}=\frac{3}{(x+1)(x^2+2)}$ $\rightarrow \quad c=1$ $a+b=0$ $b+c=0 \qquad b=-1 \quad \& \quad a=1$ $\therefore \ \frac{3}{(x+1)(x^2+2)}=\frac{1}{x+1}-\frac{x-1}{x^2+2}$	1- Correct expression 1- Correct answers	Generally well done

(ii) $\int \frac{3}{(x+1)(x^2+2)}\,dx = \int \frac{1}{x+1} - \frac{x-1}{x^2+2}\,dx = \int \frac{1}{x+1} - \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx$ $= \ln\|x+1\| - \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$ $= \ln\left\|\frac{x+1}{\sqrt{x^2+2}}\right\| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$	1-correct integrand 2- Correct integration of all 3 function	Generally well done

Question 12

(a)  (i) PQ is a focal chord through the focus (−ae,0) ∴ Tangents at P and Q will intersect on the directrix $x=-a/e$. ✓ ∴ The x-coordinate of T is $-a/e$	1m answer with reason	Many students failed to use the property of an ellipse ~~with~~ viz. tangents from the point of contact of a focal chord meet on the directrix.
(ii) $T[-a/e, y_0]$ lie on the tangent PT, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ $\Rightarrow -\frac{\cos\theta}{e}+\frac{y_0\sin\theta}{b}=1$ $\Rightarrow y_0=\frac{b}{\sin\theta}\left[1+\frac{\cos\theta}{e}\right]$ i.e. $=\frac{b}{e\sin\theta}[e+\cos\theta]$ ✓	1m correct answer in any form	As the answer in the question paper was incorrect ~~one~~ mark was given for reasonable working towards.

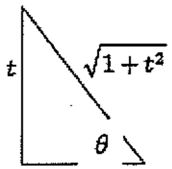
(iii) $S(ae,0)$, $P(a\cos\theta, b\sin\theta)$ gradiant of $SP = \dfrac{b\sin\theta}{a(\cos\theta - e)}$ ✓ ∴ Equation of SP is $(b\sin\theta)x - a(\cos\theta - e)y = aeb\sin\theta$ ✓	Correct gradient Correct working	majority of the students got the full mark for this.
(iv) $TM = \dfrac{\left\lvert -\frac{ab\sin\theta}{e} + \frac{ab}{e\sin\theta}(e^2 - \cos^2\theta) - abe\sin\theta \right\rvert}{\sqrt{b^2\sin^2\theta + a^2(e-\cos\theta)^2}}$ ✓ $= \dfrac{\frac{ab}{e\sin\theta}\left\lvert -\sin^2\theta + (e^2 - \cos^2\theta) - e^2\sin^2\theta \right\rvert}{a\sqrt{(1-e^2)\sin^2\theta + (e-\cos\theta)^2}}$ ✓ $= \dfrac{\frac{ab}{\sin\theta}\lvert e^2\cos^2\theta - 1)}{a(1 - e\cos\theta)}$ $= \dfrac{b(1+e\cos\theta)}{e\sin\theta}$ ✓	Correct sub. into formula working to simplify Correct answer	Full marks awarded for ~~Rea~~ reasonable working towards.

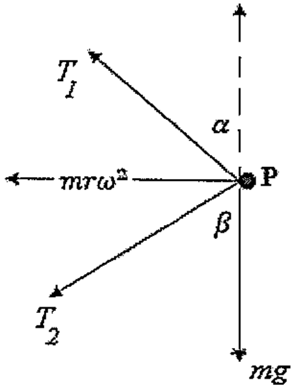
(v) $TM = TR = \frac{b(1+e\cos\theta)}{e\sin\theta}$ — ① ✓ Similarly $TN = TR = \frac{b(1+e\cos\alpha)}{e\sin\alpha}$ — ② ✓ Form ① and ② $TM = TR = TN$ $\therefore$ T is the centre of the circle that touches ~~SPQ~~ SM, SN and PQ at M, N and R respectively.	Correctly finding the equivalents Correct reasoning	

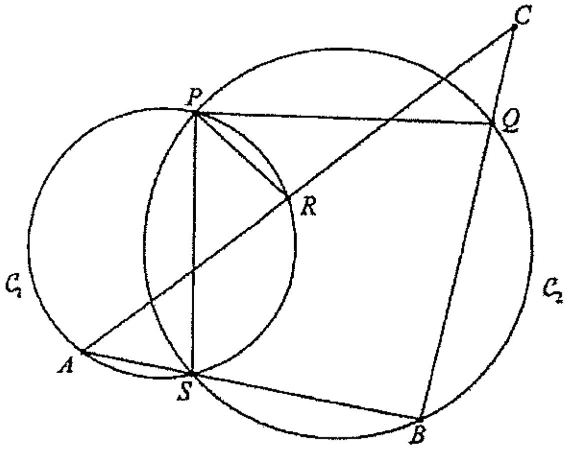
Q12 (b) $z = \cos\theta + i\sin\theta$ $\Rightarrow z^n = \cos(n\theta) + i\sin(n\theta)$ and $\bar{z}^n = \cos(n\theta) - i\sin(n\theta)$ $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$		
n=1 $\cos\theta = \frac{1}{2}(z+\frac{1}{z}) \Rightarrow \cos^6\theta = \frac{1}{64}(z+\frac{1}{z})^6$ $\sin\theta = \frac{1}{2i}(z-\frac{1}{z}) \Rightarrow \sin^6\theta = -\frac{1}{64}(z-\frac{1}{z})^6$	1 mark	a number of students failed to use this technique with De Moivre's theorem.
$\therefore \cos^6\theta - \sin^6\theta = \frac{1}{64}\left[(z+\frac{1}{z})^6 + (z-\frac{1}{z})^6\right]$		
$= \frac{1}{64}\left[2z^6 + 30z^2 + \frac{30}{z^2} + \frac{2}{z^6}\right]$	1 mark	
$= \frac{1}{64}\left[2(z^6+\frac{1}{z^6}) + 30(z^2+\frac{1}{z^2})\right]$ $= \frac{1}{64}\left[4\cos 6\theta + 60\cos 2\theta\right]$	1 mark	
$= \frac{1}{16}\left[\cos 6\theta + 15\cos 2\theta\right]$		

(c) $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$ $= \frac{1 - \ln x}{x^2}$	Correct answer	majority got this correct
$\therefore \int \frac{1 - \log x}{x \log x}\, dx = \int \frac{x}{\log x}\left(\frac{1 - \log x}{x^2}\right) dx$ $= \int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx$	1 mark	
$= \ln y + C$ $= \ln\left(\frac{\ln x}{x}\right) + C$	1 mark	

Q13 (a)

(i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t}{1+t^2}$ 	1- Using correct formulae with diagram 1- Correct working	Generally well done
(ii) $\int_0^{\frac{\pi}{4}} \frac{4}{5-3\cos 2\theta}\, d\theta$ $= \int_0^1 \frac{4}{5-\frac{3(1-t^2)}{1+t^2}} \times \frac{1}{1+t^2}\, dt$ $dt = \sec^2\theta d\theta$ $= 1 + t^2\, d\theta$ $\theta = \frac{\pi}{4} \rightarrow t = 1$ $\theta = 0 \rightarrow t = 0$ $= \int_0^1 \frac{4}{5+5t^2-3+3t^2}\, dt$ $= \int_0^1 \frac{4}{2+8t^2}\, dt = 2\int_0^1 \frac{2}{1+4t^2}\, dt$ $= [\tan^{-1} 2x]_0^1$ $= \tan^{-1} 2$	1- Correct substitution 1- Correct integrand 1- Correct answer	Generally well done
(b) $\overrightarrow{BA} = i\overrightarrow{BC}$ $\overrightarrow{OA} - \overrightarrow{OB} = i(\overrightarrow{OC} - \overrightarrow{OB})$ $\overrightarrow{OB}(i-1) = -\overrightarrow{OA} + i\overrightarrow{OC} = -1 - i + 7i - 3$ $= 6i - 4$	1- Correct rotation expression of vectors	Some students did not use correct direction of vector.

$\therefore\ \overrightarrow{OB} = \frac{6i-4}{i-1} = \frac{6i-6-4i-4}{-2} = 5-i$	1- Correct answer	
(c) 	1- Correct diagram with all labelling	Generally well done
Resolve forces vertically & horizontally: $T_1\cos\alpha - T_2\cos\beta = mg \quad (1)$ $T_1\sin\alpha + T_2\sin\beta = mr\omega^2 \quad (2)$	1- Correct expressions of forces	Generally well done
Applying the inequality $T_1 > T_2$ to both expressions (1) & (2) $T_1\cos\alpha - T_2\cos\beta = mg > T_1\cos\alpha - T_1\cos\beta \quad (3)$ $T_1\sin\alpha + T_2\sin\beta = mr\omega^2 < T_1\sin\alpha + T_1\sin\beta \quad (4)$	1- Correct expression of inequalities	
Taking reciprocal of (3) $\rightarrow \frac{1}{mg} < T_1(\cos\alpha - \cos\beta) \quad (5)$ $(5)\times(4) \quad \rightarrow \quad \frac{mr\omega^2}{mg} < \frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}$		Many had difficulty with inequality
Since $0 < \alpha < \beta < \frac{\pi}{2} \quad \rightarrow \quad \cos\alpha < cos\beta \ \ and \ \ \cos\alpha - cos\beta > 0$	1- Correct answer with reasoning	

∴ $\omega^2 < \frac{g}{r}\left(\frac{sin\alpha+sin\beta}{cos\alpha-cos\beta}\right)$.		
(d) 		
$< PRA = < PSA$ (Angles at circumference subtended by same arc) (1)		
$< PSA = < PQB$ (Exterior angle is equal to interior opposite angle in cyclic quadrilateral)	1- with reasoning	Generally well done
∴ $< PRA = < PQB$	1- with reasoning	
(ii) $< PRC = 180^o - < PRA$ (Straight angle) $< PSB = 180^o - < PSA$ ∴ $< PRC = < PSB$ (using (1))	1- with reasoning	
But $< PSB = < PQC$ (Exterior angle is equal to interior opposite angle in cyclic quadrilateral) ∴ $< PRC = < PQC$		
∴ PRQC is cyclic quadrilateral (Two points lie on the same sides of an interval, and the angles subtended at these points by the interval are equal)	1- with reasoning	Most students did not get full mark because they did not write the correct theorem. This is the reverse theorem of angles in the alternate segment.

Question 14

14(a) $y = (x-2)(6-x)$

$= -x^2 + 8x - 12$

axis of symmetry of the parabola is

$x = \frac{-8}{-2} = 4$

1 mark

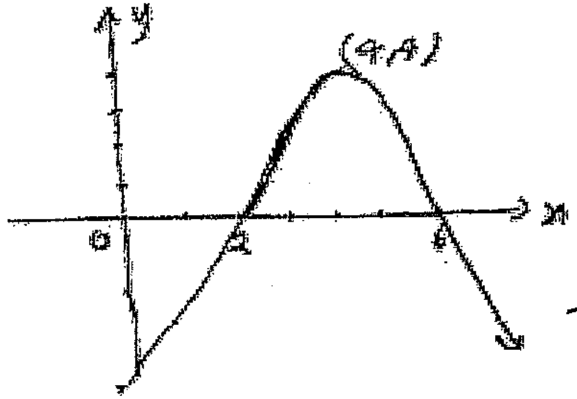
when $x = 4$, $y = (4-2)(6-4)$

$= 4$

$\therefore$ The turning point is $(4,4)$

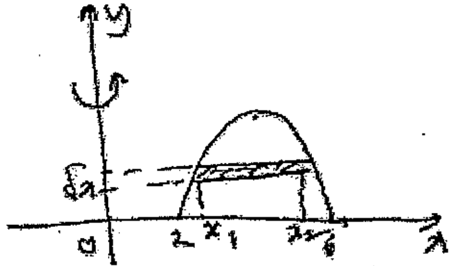
1 mark

majority got this correct

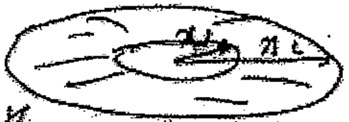


(b)

(i)



Annulus formed when the strip shaded is rotated about the y axis



Area of the annulus $\delta A = \pi(x_2^2 - x_1^2) \longrightarrow$ 1m

$y = -x^2 + 8x - 12$

$x^2 - 8x + (y+12) = 0$

$x_2 = 4 + \sqrt{4-y}$

$x_1 = 4 - \sqrt{4-y}$

1m

$= \pi(x_2+x_1)(x_2-x_1)$

$= \pi(8)(2\sqrt{4-y})$

$= 16\pi\sqrt{4-y}$

1m

(ii) Volume of the annulus $\delta V = \delta A \cdot \delta x = 16\pi\sqrt{4-y}\cdot\delta x$

volume of the solid formed $V = \sum_{\delta x \to 0} \delta V = \int_0^4 16\pi\sqrt{4-y}\,dx$

$= 16\pi\left[-2\frac{(4-y)^{3/2}}{3}\right]_0^4$

$= \frac{256}{3}\pi$ units^3

Correct integral (1m)

correct (1m) answer

Students need to draw the annulus with measurements and show working

(c)(i) $I_n = \int_0^3 x^n\sqrt{9-x^2}\,dx$

$= \int_0^3 \frac{x^{n-1}}{-2}\cdot(-2x)\sqrt{9-x^2}\,dx$ — 1m

$= \underbrace{\left[-\frac{x^{n-1}}{2}\cdot\frac{2}{3}(9-x^2)^{3/2}\right]_0^3}_{0} + \int \frac{2}{3}(9-x^2)^{3/2}\cdot\left(\frac{n-1}{2}\right)x^{n-2}\,dx$

majority got this correct

$= \frac{(n-1)}{3}\int_0^3 x^{n-2}(9-x^2)\sqrt{9-x^2}\,dx \longrightarrow$ 1m

$I_n = \frac{(n-1)}{3}\left[9I_{n-2} - I_n\right]$

$\Rightarrow$ ~~$I_n = \frac{(n-1)}{3}\left(1+\frac{n-1}{3}\right)I_n = \frac{3(n-1)}{3}I_{n-2}$~~

$\left(1+\frac{n-1}{3}\right)I_n = 3(n-1)I_{n-2}$

1 mark

14 c (ii) $I_n = \frac{9(n-1)}{(n+2)} I_{n-2}$

$\int_0^3 x^5 \sqrt{9-x^2}\,dx = I_5$

$= \frac{36}{7} I_3$ ✓

$= \frac{36}{7} \cdot \frac{18}{5} I_1$

$= \frac{36}{7} \times \frac{18}{5} \times 9$

$= \frac{5832}{35}$ ✓

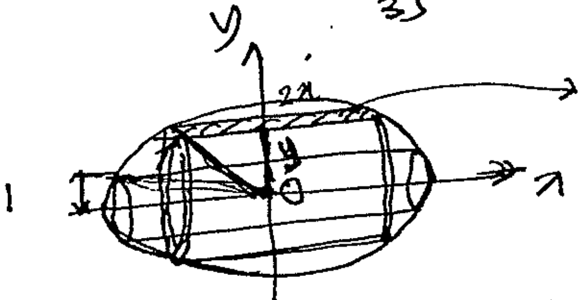
$I_1 = \int_0^3 x\sqrt{9-x^2}\,dx$

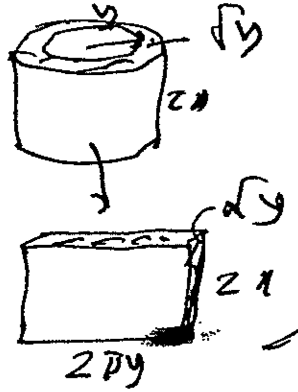
$= -\frac{1}{3}\left[(9-x^2)^{3/2}\right]_0^3$

$= 9$

2 marks.

(d)





Vol. of the shell $\delta V = 2\pi y \cdot 2x \cdot \delta y$

$= 5\pi y\sqrt{16-y^2}\,\delta y$

∴ Vol. of solid formed $= \Sigma \delta V$

$= \int_1^4 5\pi y\sqrt{16-y^2}\,dy$

$= 5\pi\left[-\frac{(16-y^2)^{3/2}}{3}\right]_1^4$

~~$= \frac{320}{3}\pi$ units^3~~

$= 25\sqrt{15}\,\pi$ units^3

— 1 m

— 1 m

1 m

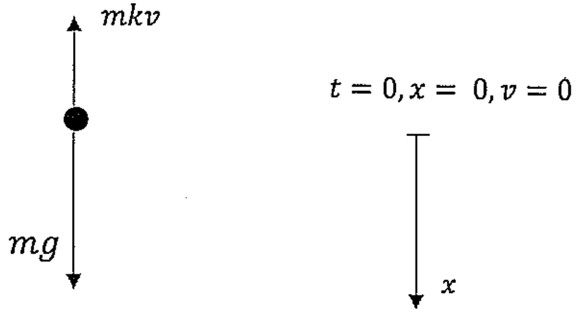
need to show the calc. of the vol. of the shell

Students need to show the shell and calculation of δV.

Many students struggled with this question and calculated ~~to~~ ~~[illegible]~~ the volume of wrong solids

Question 15

(i) $f'(x) = e^x + \frac{1}{e^x} > 0 \quad e^x > 0$ $\therefore$ $f(x)$is an increasing function for all x	1- Correct differentiation & conclusion	
(ii) $x = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$ $\rightarrow \quad e^{2y} - xe^y - 1 = 0$		Generally well done
$e^y = \frac{x \pm \sqrt{x^2+4}}{2} \quad \rightarrow \quad y = \log_e\left(\frac{x \pm \sqrt{x^2+4}}{2}\right)$	1-correct formula for y	
Since $\sqrt{x^2+4} > x$ and $\frac{x \pm \sqrt{x^2+4}}{2} > 0$ (domain of $\log_e x$) $\rightarrow \quad y = \log_e\left(\frac{x+\sqrt{x^2+4}}{2}\right)$	1-correct answer with reasoning	
(iii) Using result from (ii) $x = \log_e\left(\frac{5+\sqrt{25+4}}{2}\right) = 1.65$	1-correct answer.	
(b)		Well done
(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ $= 0 - (-A) = A$ $\rightarrow \alpha^2 + \beta^2 = A - \alpha^2 \quad (1)$	1-correct answer with working	
(ii) $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{A-\alpha^2}{\beta\gamma}$ (Multiplying by α) $= \frac{A\alpha-\alpha^3}{\alpha\beta\gamma} = \frac{A\alpha-\alpha^3}{\frac{2}{2}} = A\alpha - \alpha^3$	2-correct answer with working	Generally well done
(iii) Let $x = A\alpha - \alpha^3$ and $P(\alpha) = 2\alpha^3 - A\alpha - 2 = 0$ $\rightarrow \quad -(A\alpha - \alpha^3) + \alpha^3 - 2 = 0$		

$\therefore\ -x+\alpha^3-2=0 \quad \rightarrow \quad x=\alpha^3-2 \quad or \quad \alpha^3=x+2$ $\therefore$ the cubic is $P(x)=2(x+2)-A(x+2)^{\frac{1}{3}}-2=0$ $2(x+1)=A(x+2)^{\frac{1}{3}} \quad \rightarrow \quad 8(x+1)^3-A^3(x+2)=0$	2-correct answer with working.	Only 2 students found the correct answer
(c)  Resultant force acting on the body: $mg-mkv=m\ddot{x}$		Well done
$\ddot{x}=k\left(\frac{g}{k}-v\right)$ when $mkv=mg \quad \ddot{x}\rightarrow 0 \quad v\rightarrow\frac{g}{k}$ Hence the terminal velocity is $V=\frac{g}{k}$ $\therefore\ \ddot{x}=k(V-v)$	1-correct formula for $\ddot{x}$ with reasoning.	
(ii) $\frac{dv}{dt}=k(V-v)$ $-k\frac{dt}{dv}=\frac{-1}{V-v} \quad \rightarrow \quad -kt=\ln\{A(V-v)\},$ A constant $t=0, v=0 \quad \rightarrow \quad A=\frac{1}{V}$	1-correct integration for t	Generally well done

$\therefore\quad -kt = \ln\left(\frac{V-v}{V}\right)\quad (1)$ $t = \frac{1}{k}\ln\left(\frac{V}{V-v}\right)$	1-correct formula for t	
The particle obtain 50% of terminal velocity when $v = \frac{1}{2}V$, Sub. into t $\rightarrow\quad T = \frac{1}{k}\ln\left(\frac{V-\frac{1}{2}V}{V}\right)\quad\rightarrow\quad T = \frac{1}{k}ln2$	1-correct answer for t	
Alternately: $\int -kdt = \int_{0}^{\frac{V}{2}} \frac{dv}{V-v}\quad\rightarrow\quad -kt = \left[\ln(V - \frac{V}{2}\right]_{0}^{\frac{V}{2}}$ $\therefore\quad -kt = \ln\left(\frac{V}{2}\right) - lnV\quad\rightarrow\quad -kt = \ln\left(\frac{1}{2}\right)\quad\rightarrow\quad T = \frac{1}{k}ln2$		
(iii) For the distance fallen: $-kt = \ln\left(\frac{V-v}{V}\right)$ from (1) $Ve^{-kt} = V - v$ $\rightarrow\quad v = V - Ve^{-kt}$		
$\frac{dx}{dt} = V - Ve^{-kt}$	1-Correct formula for $\frac{dx}{dt}$	Many students use the formula $\ddot{x} = k\left(\frac{g}{k} - v\right)$, so could not find answer in terms of k, V & t
$x = Vt + \frac{V}{k}e^{-kt} + C$ when $t = 0, x = 0\quad\rightarrow\quad C = -\frac{V}{k}$ $\therefore\quad x = Vt + \frac{V}{k}e^{-kt} - \frac{V}{k},$	1-correct answer for x	

Question 16

(a) (i) $k\overrightarrow{BA} = \overrightarrow{BC}\,\text{cis}\,\theta$

$k\overrightarrow{VU} = \overrightarrow{VW}\,\text{cis}\,\theta$ where k is a constant

→ 1 mark

$$\therefore \frac{\overrightarrow{BA}}{\overrightarrow{VU}} = \frac{\overrightarrow{BC}}{\overrightarrow{VW}}$$

$$\frac{a-b}{u-v} = \frac{c-b}{w-v}$$

→ 1 mark

$\Rightarrow (a-b)(w-v) = (c-b)(u-v)$

$\Rightarrow aw - av - bw + \cancel{bv} = cu - cv - bu + \cancel{bv}$

→ 1 mark

$\Rightarrow aw + bu + cv = av + bw + cu$

(ii) when ΔABC is equilateral

$|a-b| = |c-b|$

$\therefore (a-b) = (c-b)\,\text{cis}\,\pi/3$ — ①

Similarly $(c-a) = (b-a)\,\text{cis}\,\pi/3$ — ②

→ 1 mark

① ÷ ② $\dfrac{a-b}{c-a} = \dfrac{c-b}{b-a}$

$\Rightarrow -(a-b)^2 = (c-a)(c-b)$

$\Rightarrow -a^2 + 2ab - b^2 = c^2 - ac - bc + ab$

→ 1 mark

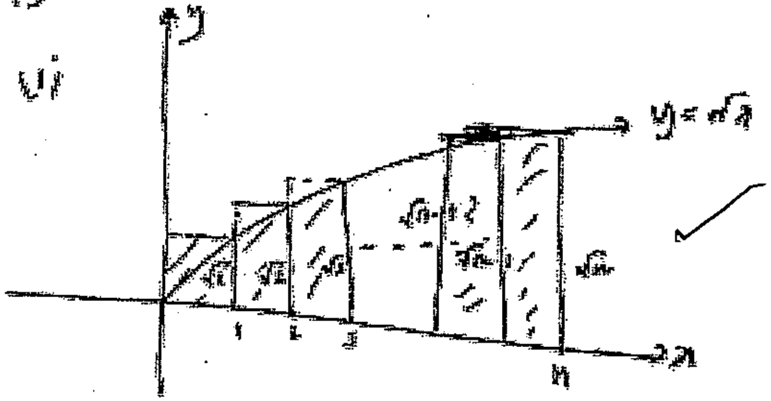
$\Rightarrow a^2 + b^2 + c^2 = ab + bc + ca$

Many students had the misconception

$$\frac{|z_1|}{|z_2|} = \frac{|z_3|}{|z_4|} \Rightarrow \frac{z_1}{z_2} = \frac{z_3}{z_4}$$

16 (b)

(i)



✓

$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$ represents the sum of the areas of the rectangles as shown above. ✓

1m

(ii) Sum of the areas of the rectangles shown > area under the curve from $x=0$ to $x=n$ — 1m

$$\Rightarrow \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n} > \int_0^n \sqrt{x}\,dx$$

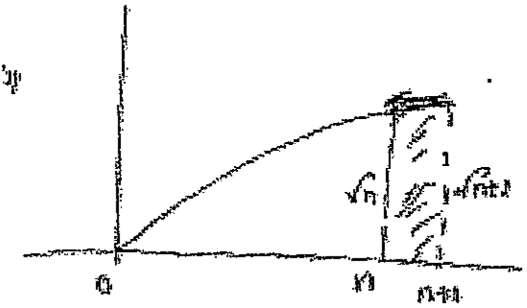
$$= \left[\frac{2x^{3/2}}{3}\right]_0^n$$

$$= \frac{2n\sqrt{n}}{3}$$ //

1m

Part (i) was not clearly answered

(iii)



area under the curve from 0 to n + area of the shaded ~~rectangle~~ trapezium

< area under the curve from 0 to (n+1) ✓ 1 m

$$\Rightarrow \frac{2n\sqrt{n}}{3} + \left[\frac{\sqrt{n+1}+\sqrt{n}}{2}\right] < \frac{2(n+1)\sqrt{n+1}}{3}$$

$$\Rightarrow \left(\frac{2n}{3}+\frac{1}{2}\right)\sqrt{n} < \left[\frac{2(n+1)}{3}-\frac{1}{2}\right]\sqrt{n+1}$$

$$\frac{(4n+3)\sqrt{n}}{6} < \left(\frac{4n+1}{6}\right)\sqrt{n+1}$$

$$\Rightarrow (4n+3)\sqrt{n} < (4n+1)\sqrt{n+1} \;//$$

1 m

Many students experienced difficulty with this question

16 (c) PERSEVERE

(i) No 2 E's together

$- P - R - S - V - R -$

There are 6 places for 4 E's to fill
The no of choices $= {}^6C_4$
P R S V R can be arranged in $\frac{5!}{2!}$ ways } 1 m

$\therefore$ The required no of arrangements

$= {}^6C_4 \times \frac{5!}{2!}$

$= \underline{\underline{900}}$ $\longrightarrow$ 1 m

(ii) Must start and end with S and P with no 2 E's together

$\underline{S}\_R\_V\_R\_P$

4 places for 4 E's — ${}^4C_4 = 1$ way ~~1m~~
R V R can be arranged in $\frac{3!}{2!} = 3$ ways $\longrightarrow$ 1 m
S and P can swap $\underline{= 2 \text{ ways}}$

$\therefore$ Total $\underline{\underline{6}}$ ways $\longrightarrow$ 1 m